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GROUPS WITH THREE REAL VALUED IRREDUCIBLE CHARACTERS*

BY

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ABSTRACT

We obtain strong restrictions on the structure of a Sylow 2-subgroup of a group with at most three real valued irreducible characters. This extends results of Iwasaki, who studied groups with at most two real valued irreducible characters.

1. Introduction

S. Iwasaki [3] proposed to study the structure of a finite group G according to its number of real valued irreducible characters. Being groups of odd order the groups with exactly one irreducible real character, in [3] he characterized the finite groups with two real valued characters. In particular, he proved that they have a normal Sylow 2-subgroup that is either homocyclic or a Suzuki 2-group of type A (see Definition VIII.7.1 of [1] for a definition). The goal of this note is to extend these results to groups with at most three real valued characters.

Theorem A: *Let G be a finite group with at most three irreducible real valued characters. Then G has a cyclic Sylow 2-subgroup or a normal Sylow 2-subgroup*

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which is homocyclic, quaternion of order 8 or an iterated central extension of a Suzuki 2-group whose center is an elementary abelian 2-group.

In other words, we cannot get the same situation as in Iwasaki's theorem, but we show that the structure of our groups is pretty similar. Now it is not possible to assure that G has a normal Sylow 2-subgroup, as the symmetric group S_3 shows. Also, we cannot rule out the quaternion group of order 8 as a possible Sylow 2-subgroup, as $\mathrm{SL}(2,3)$ shows. In Section 3 we will also give examples that show that there are groups with 3 real valued irreducible characters whose Sylow 2-subgroup is a Suzuki group which is not of type A and also groups whose Sylow 2-subgroup is a central extension of a Suzuki 2-group.

It is worth mentioning that if we allow 4 real valued characters then the group G does not need to be solvable any more: the group $\mathrm{PSL}(2,7)$ is a nonsolvable group with exactly 4 real valued characters.

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2. Proof of Theorem A

We begin by stating several recent results on real and rational valued characters that we will use. Recall that $o(\theta)$ is the determinantal order of the character θ. In general, we use the notation in [2].

LEMMA 2.1: *Let N be a normal subgroup of a group G and let $\theta \in \mathrm{Irr}(N)$ be G-invariant, real of odd degree. Suppose that $o(\theta) = 1$. Then θ has a real extension to G.*

Proof. See Theorem 2.3 of [5] ∎

LEMMA 2.2: *Let S be a nonabelian simple group. Then S has at least 4 real elements of pairwise different orders.*

Proof. This follows from Theorem 3.1 of [4], for instance. ∎

The proof of this result relies on the classification of finite simple groups. We will, however, apply it only to simple groups S with just one orbit of involutions under the action of $\mathrm{Aut}(S)$, so this is the only part of the classification of simple groups that we will be using.

Recall that an element $x \in G$ is real if x and x^{-1} are G-conjugate. By Brauer's lemma on character tables (Theorem 6.32 of [2]), we know that the number of irreducible real characters of G is the number of conjugacy classes of G consisting of real elements.

We need the following elementary but useful lemma.

Lemma 2.3: *Let G be a group with three real valued irreducible characters. Then G has at most two conjugacy classes of involutions. Furthermore, if G has two conjugacy classes of involutions then, together with the identity, they form a normal subgroup of G.*

Proof. First, we note that any class of involutions is a real conjugacy class. Also, the class of the identity is a real conjugacy class. Hence, the first part of the lemma easily follows.

Now assume that G has two conjugacy classes of involutions K_1 and K_2. Let $x, y \in G$ be two involutions. Then

$$(xy)^y = yx = (xy)^{-1},$$

so xy belongs to a real class. But we know that all the real elements belong to $\{1\} \cup K_1 \cup K_2$. We have proved that this subset of G is a subgroup and the lemma follows ∎

Observe also that this normal subgroup is an elementary abelian 2-group.

Finally, we need the following elementary result.

Lemma 2.4: *Assume that a group X of an odd order acts on an elementary abelian 2-group V. Then the number of orbits of X on V is even. In particular, X cannot act on V with two orbits of nonidentity elements.*

Proof. It suffices to observe that any orbit of X on V has odd size and that $|V|$ is even. ∎

In the next result we already find strong restrictions for the structure of groups with at most three real valued characters.

Theorem 2.5: *Let G be a group with at most three real valued irreducible characters. Then G is a solvable group of 2-length one whose Sylow 2-subgroup is homocyclic, quaternion of order 8 or an iterated central extension of a Suzuki 2-group whose center is an elementary abelian 2-group.*

Proof. We start by proving that a group G with at most three real valued irreducible characters is solvable. We argue by induction on $|G|$. If N is any minimal normal subgroup of G, then G/N is solvable by the inductive hypothesis. Thus we may assume that N a direct product of copies of a nonabelian simple group. If G has one or two real irreducible characters, we already know that G is solvable. So we may assume that G has exactly three real valued irreducible characters. Now, it follows from Lemma 2.3 that G has exactly one conjugacy class of involutions. This implies that N is a nonabelian simple group. Since $Z(N) = 1$, we deduce that $C_G(N) = 1$, again by applying induction in the group $G/C_G(N)$. Thus G is an almost simple group with exactly one conjugacy class of involutions. It follows from Lemma 2.2 that G has at least 4 real conjugacy classes and by Brauer's lemma on character tables, 4 irreducible real valued characters. This proves that G is solvable.

Next, we prove that the 2-length of G is at most one. (That is to say: we prove that G has an odd order normal subgroup R such that G/R has a normal Sylow 2-subgroup.) We argue by induction on $|G|$. We certainly may assume that $O_{2'}(G) = 1$. Now, let N be a minimal normal subgroup of G, which is an elementary abelian 2-group. By the inductive hypothesis, G/N has two length at most one. Let L and M be the normal subgroups of G such that $L/N = O^{2'}(G/N)$ and $M/N = O^2(L/N)$. Now we have that M/N is a 2'-group and we claim that we may assume that $O^2(M) = M$. Otherwise, since N is a minimal normal subgroup of G, we would have that $O^2(M)$ is a 2'-group and $M = O^2(M) \times N$. But since $O_{2'}(G) = 1$, this implies that $M = N$ and G has 2-length one.

Now take $\lambda \in \mathrm{Irr}(N)$ nonprincipal and let T be the inertia group of λ in M. Since M/N is a 2'-group, there exists a canonical extension $\hat{\lambda}$ of λ to T (see Corollary 8.16 of [2]). In particular, $\hat{\lambda}$ is real valued by the uniqueness of canonical extensions. Now $\theta = \hat{\lambda}^M \in \mathrm{Irr}(M)$ is a real valued character of odd degree. Recall that for a real valued character θ, $o(\theta) \leq 2$. (See [5].) Since $O^2(M) = M$, we have that $o(\theta) = 1$. Now, we can apply Lemma 2.1 to deduce that θ has a real valued extension $\tilde{\theta}$ to its inertia group in G. Now $\tilde{\theta}^G \in \mathrm{Irr}(G)$ is a real valued character. We deduce that G/N has at most two real valued characters. If G/N has odd order, then G has a normal Sylow 2-subgroup and we are done. Otherwise, G/N has exactly two irreducible real valued characters and by Iwasaki's theorem, we have that G/N has a normal Sylow 2-subgroup. Thus G has a normal Sylow 2-subgroup, as desired.

Finally, we want to prove that the Sylow 2-subgroup of G is homocyclic, quaternion of order 8, or an iterated central extension of a Suzuki 2-group whose center is an elementary abelian 2-group. As before, we may assume that $O_{2'}(G) = 1$, so that G has a normal Sylow 2-subgroup P. Also, we may assume that G has exactly three real valued irreducible characters by Iwasaki's theorem.

Now, let X be a Hall 2-complement of G. Assume first that the Sylow 2-subgroup P of G has exactly one involution. Then it is cyclic or generalized quaternion. If it is a generalized quaternion group of order bigger than 8, then it does not have any nontrivial odd order automorphism, and we deduce that G is a quaternion group. But then it has at least 4 rational characters. This contradiction means that if P is generalized quaternion, then it has order 8.

Now we may assume that P has more than one involution. If G has exactly one class of involutions then the result follows from a deep theorem of Thompson (see Theorem IX.8.6 of [1]). Hence, by Lemma 2.3 we may assume that G has exactly two conjugacy classes of involutions and that together with the identity, they form a normal subgroup V of G. Assume first that V is central in P. Then a Hall 2-complement of G acts on V with two orbits of nonidentity elements. This contradicts Lemma 2.4.

Let $W = V \cap Z(P)$. We deduce that $1 < W < V$ is a normal subgroup of G. By induction, the Sylow 2-subgroup P/W of G/W is quaternion of order 8, homocyclic or an iterated central extension of a Suzuki 2-group whose center is an elementary abelian 2-group.

Assume first that P/W is an iterated central extension of a Suzuki 2-group whose center Z/W is an elementary abelian 2-group. The group G/W has at most two conjugacy classes of involutions, and we deduce that the Hall 2-complement of G/W acts on Z/W with only one nontrivial orbit. Since $Z \cap V > W$, we deduce that $Z = V$. It is also clear that if P/W is a quaternion group, then $Z(P/W) = V/W$.

In both cases we have that $Z(P) \leq V$ so that $Z(P) = W$. Now if P/W is an iterated central extension of a Suzuki 2-group, then the same happens with P. It is well-known that the quaternion group of order 8 is not capable, so P/W cannot be isomorphic to Q_8.

Assume now that P/W is homocyclic. Remember also that $V/W = \Omega_1(P/W)$ (the number of conjugacy classes of involutions of G/W cannot be bigger than one by the previous lemma and our hypothesis) and $\Omega_1(P) = V$ is an elementary abelian 2-group. Observe also that $P' \leq W = V \cap Z(P)$.

Then for any element x in $P - W$ there exists some power of x in $V - W$. It follows that x does not belong to the center of P, so again $Z(P) = W$. In particular, P has class 2.

If P/W is elementary abelian, then $P = V$ and $Z(P) = V > W$. This is a contradiction. Hence, there exists $x \in P$ such that $x^2 \in V - W$. Now, $[x^2, y] = [x, y]^2 = 1$ for every $y \in G$. (The second equality follows from the fact that $P' \leq W$.) This shows that $x^2 \in Z(P) = W$. This contradiction shows that P/W cannot be homocyclic. ∎

Finally, the next result completes the proof of Theorem A.

THEOREM 2.6: *Suppose that G is a finite group with exactly three real valued irreducible characters. Let $P \in \mathrm{Syl}_2(G)$. Then P is cyclic or normal in G.*

Proof. We argue by induction on $|G|$. We already know that G is solvable with 2-length 1. In particular, we may assume that $O_{2'}(G) > 1$. Let V be a minimal normal subgroup of G of odd order. Either by induction or by the two real characters case, we have that $PV \triangleleft G$. If $[P, V] = 1$, then $P \triangleleft G$, and we are done. Thus $C_P(V) < P$. Let $tC_P(V) \in P/C_P(V)$ of order 2, and let $v \in V$ such that $1 \neq v^{-1}v^t = w$. Then w is inverted by t, and the class of w is real. Hence, the unique non-trivial real classes of G are the class of involutions and the class of w. Suppose that $N = O_2(G) > 1$. Hence $N \subseteq P$ and thus $N \cap Z(P) > 1$. Let $s \in N \cap Z(P)$ of order 2. Then

$$(sw)^t = sw^{-1} = w^{-1}s = (sw)^{-1}$$

and this is impossible.

Now, if $x \in P$ is an involution, we have that

$$V = [V, x] \times C_V(x)$$

and x inverts $[V, x] > 1$. Hence, G has odd order real elements, and hence, the real classes are this class of odd real elements, the (unique) class of involutions of G and the identity.

Since x inverts some element in V, there exists $\lambda \in \mathrm{Irr}(V)$ such that $\lambda^x = \overline{\lambda}$. Let T be the inertia subgroup of λ in PV and θ the canonical extension of λ to T. By uniqueness, $\theta^x = \overline{\theta}$. Hence

$$\eta = \theta^{PV} = (\theta^x)^{PV} = \overline{\theta}^{PV} = \overline{\theta^{PV}}$$

is a real valued irreducible character of PV. By Corollary 2.2 of [5], we conclude that G has an irreducible real character not having V in its kernel. Thus G/V has exactly two real characters. Hence P is homocyclic or Suzuki. Also, if K/V is the set of elements of G/V with $x^2 = 1$, then K/V is a normal abelian subgroup of G/V. Also, $K = IV$, where $I = \Omega_1(Z(P))$.

Now, G/V acts on V. Among all involutions x in I we choose x such that $[V, x]$ is as large as possible. Write $V = W \times C_V(x)$, where $W = [V, x]$ is inverted by x. Now, let $y \in I$ be any other involution of G. Hence $xy = yx$. In particular, W is normalized by y. Suppose that $1 \neq w \in W$ is centralized by y. Then

$$(wy)^x = w^{-1}y = yw^{-1} = (wy)^{-1},$$

is a real element, and this is impossible. Hence $C_W(y) = 1$. In particular, $W = [W, y] \subseteq [V, y]$. We conclude that $W = [V, x] = [V, y]$ for all $y \in I$.

Now, if $g \in G$, we have that $x^g = vz$ for some $z \in I$ and $v \in V$. Hence

$$W^g = [V, x^g] = [V, vz] = [V, z] = [V, x] = W,$$

and we conclude that $W \triangleleft G$. Hence $W = V$ and $C_V(x) = 1$ and all elements of V are inverted by any involution of G.

Suppose that $x, y \in P$ are different involutions of P. Then $xy \neq 1$ is an involution of G. If $v \in V$, then

$$v^{-1} = v^{xy} = (v^x)^y = v$$

and this is a contradiction. Hence, P has a unique involution. Thus P is cyclic or generalized quaternion. So it is cyclic. ∎

3. Examples

As we have mentioned in the introduction, there are examples of groups with three real valued characters whose Sylow 2-subgroup is a Suzuki group which is not of type A. Let P be the Sylow 2-subgroup of $\mathrm{PSU}(3,4)$. This is a Suzuki 2-group of order 64 with $P' = Z(P)$ of order 4. This group has an automorphism τ of order 15. Consider the semidirect product $G = P\langle\tau\rangle$. Observe that $\langle\tau\rangle$ acts faithfully and transitively on the nontrivial elements of P/P' and transitively on $P' - \{1\}$. It is easy to see that the only real characters of G are the principal character, λ^G, where λ is any linear character of P and $\hat{\theta}^G$, where $\hat{\theta}$ is the canonical extension of any nonlinear character θ of P to a subgroup of G of index 3.

Next, we construct an example of a group G with three real valued characters whose Sylow 2-subgroup is a central extension of a Suzuki 2-group. The Suzuki 2-group of type A and size 2^6 has a central extension P of order 2^7 and exponent 4. This group has an automorphism τ of order 7 that acts transitively on both the nontrivial elements of $P/Z_2(P)$ and the nontrivial elements of $Z_2(P)/Z(P)$. Also, any character of P whose kernel does not contain $Z(P)$ has degree 8 (in particular, there is a unique such character and hence it is rational valued) and, of course, the nonlinear characters of $P/Z(P)$ have degree 2. Let $G = P \rtimes \langle \tau \rangle$. The group $G/Z(P)$ is one of the groups with two real valued characters that Iwasaki considered. The group G has exactly one more real valued irreducible character: the canonical extension of the irreducible character of P of degree 8.

Finally, we remark that we do not know whether or not it is possible to erase the word "iterated" in the statement of Theorem A.

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